STATISTICAL REASON FOR THE 1.5σ SHIFT

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INTRODUCTION

Motorola, Inc. introduced its 6σ quality initiative to the world in the 1980s. Almost since that time, quality practitioners have questioned why followers of this philosophy insist on adding a shift to the average before estimating process capability. According to published six-sigma literature[1], this shift should range from 1.4σ to 1.6σ, with 1.5σ recommended for most processes. When asked the reason for such an adjustment, six-sigma advocates claim it is necessary, but offer only personal experiences and three dated empirical studies[2, 3, 4] as justification (two of these studies are 25 years old, the third is 50).

By examining the sensitivity of control charts to detect changes of various magnitudes, this article provides a statistical basis for including a shift in the average that is dependent on the chart's subgroup size.

PROCESSES ARE DYNAMIC

According to the second law of thermodynamics, entropy is constantly increasing in all systems, which eventually returns them to their most random state. Applied to manufacturing, this concept means that right after a process is stabilized, entropy immediately goes to work to make it unstable[5]. Fluctuations in inputs, as well as deterioration of its own elements, subject every process to an almost continual barrage of changes. Some of these will push the process average higher, while others will move it lower.

But isn't this one of the main reasons for using control charts? When the average undergoes one of these inevitable shifts, the chart alerts us to the change. Corrective action is then initiated to return the average to its original value, thus restoring process stability.

An ideal scenario but, unfortunately, a control chart will not detect every movement in the process average. To understand why not, we need to review a few charting fundamentals.

CONTROL CHART MECHANICS

An elliptical-shaped lamination used in military aircraft is formed by pressing a raw material into the proper shape and thickness. Being a critical characteristic, thickness is to be monitored with an $\overline{X}$, R control chart. Choosing a subgroup size, *n*, of four, we collect this many consecutively produced laminations every hour, measure their thickness, and then calculate the average and range of these measurements.

To evaluate process stability, the subgroup average is plotted on the $\overline{X}$ chart, while the subgroup range is plotted on the R chart, as is illustrated in Figure 1. When subgroup statistics from at least 20 subgroups are plotted, the $\overline{X}$ s are summed, then averaged to obtain $\overline{\overline{X}}$. In a similar manner, the subgroup ranges are averaged to derive $\overline{R}$. In the following formulas, *k* represents the number of subgroups plotted on the chart.

$$\overline{\overline{X}} = \frac{\sum_{i=1}^{k} \overline{X}_i}{k} = 80.0 \qquad \overline{R} = \frac{\sum_{i=1}^{k} R_i}{k} = 4.12$$

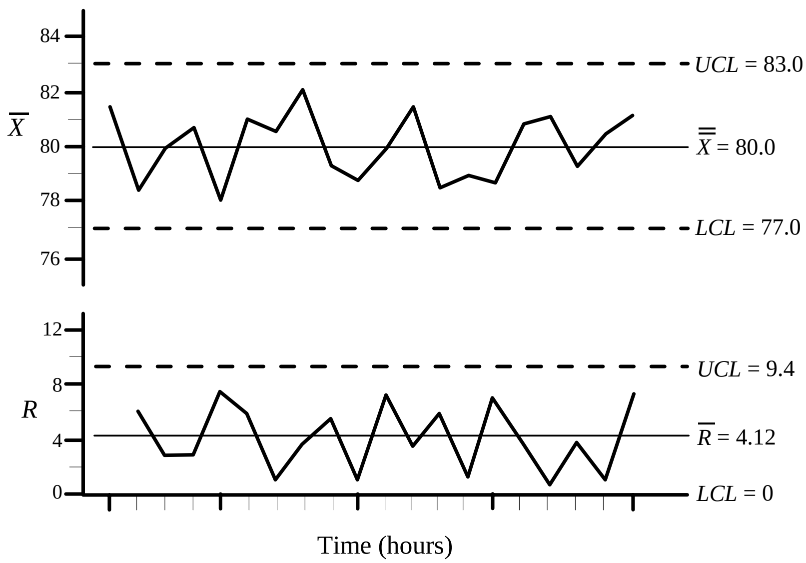

Figure 1

$\overline{\overline{X}}$ becomes the centerline of the $\overline{X}$ chart while $\overline{R}$ serves as the centerline of the range chart. The upper (*UCL*) and lower (*LCL*) control limits are determined from the next formulas, where A_2, D_3, and D_4 are constants for a subgroup size of four[6]. If all *k* subgroups are in control, as they are in Figure 1, these limits become the ongoing control limits for monitoring lamination thickness.

$$UCL_{\overline{X}} = \overline{\overline{X}} + A_2 \overline{R} = 80.0 + 0.73(4.12) = 83.0$$
$$LCL_{\overline{X}} = \overline{\overline{X}} - A_2 \overline{R} = 80.0 - 0.73(4.12) = 77.0$$

$$UCL_R = D_4 \overline{R} = 2.28(4.12) = 9.4$$
$$LCL_R = D_3 \overline{R} = 0.0(4.12) = 0.0$$

THE DISTRIBUTION OF LAMINATION THICKNESS

Because the chart indicates good process stability, the average, μ, and standard deviation, σ, of the output distribution for lamination thickness may be estimated via these equations, where d_2 is a constant based on the subgroup size[6]. The "^" symbol appearing above a process parameter indicates that this quantity is an estimate of the true value of that parameter.

$$\hat{\mu} = \overline{\overline{X}} = 80.0 \qquad \hat{\sigma} = \frac{\overline{R}}{d_2} = \frac{4.12}{2.06} = 2.0$$

Comparing $\hat{\mu}$ to the desired average thickness provides an indication of the ability of this process to produce parts on target. With a target of 80.0, the process is correctly centered.

$\hat{\sigma}$ furnishes information about the variation in individual thickness measurements around their average. Assuming the output for thickness is close to a normal distribution, approximately 99.73 percent of these measurements will be between $\hat{\mu}$ minus $3\hat{\sigma}$ and $\hat{\mu}$ plus $3\hat{\sigma}$. As is shown in Figure 2, this interval extends from 74 (80 – 3 × 2) to 86 (80 + 3 × 2).

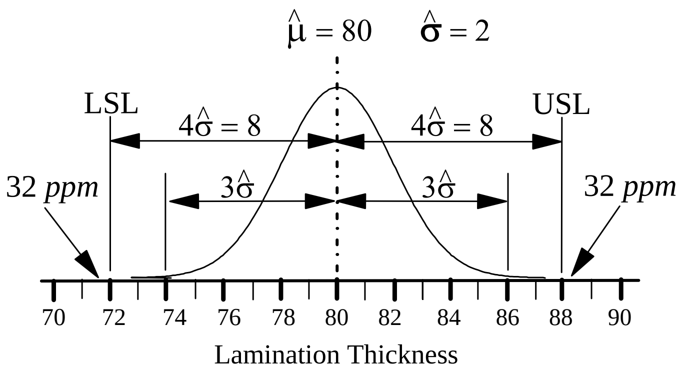

Figure 2 The output distribution of lamination thickness.

Given specification limits of 80.0 ± 8.0 for thickness, the C_{PK} index of process capability is estimated as 1.33. Here, USL is the upper specification limit while LSL is the lower specification limit.

$$\begin{aligned}\hat{C}_{PK} &= \text{Minimum}\left(\frac{\hat{\mu} - \text{LSL}}{3\hat{\sigma}}, \frac{\text{USL} - \hat{\mu}}{3\hat{\sigma}}\right) \\ &= \text{Minimum}\left(\frac{80.0 - 72.0}{3(2.0)}, \frac{88.0 - 80.0}{3(2.0)}\right) \\ &= \text{Minimum}(1.33, 1.33) = 1.33\end{aligned}$$

A C_{PK} of 1.33 for a process centered at the middle of its tolerance means the process average is 4.0σ (3 × 1.33) away from either specification limit (Figure 2). At this distance, only 32 out of one million laminations (32 *ppm*) will be thinner than the LSL of 72, with a like amount thicker than the USL of 88, for a total of 64 *ppm*.

THE DISTRIBUTION OF SUBGROUP AVERAGES

The top portion of Figure 3 displays the process output for individual thickness measurements, assuming it has a normal distribution with μ equal to 80 and σ equal to 2. Shown directly below this is the distribution of subgroup averages that are plotted on the $\overline{X}$ chart. Thanks to the Central Limit Theorem, we know that these $\overline{X}$ s have close to a normal distribution.

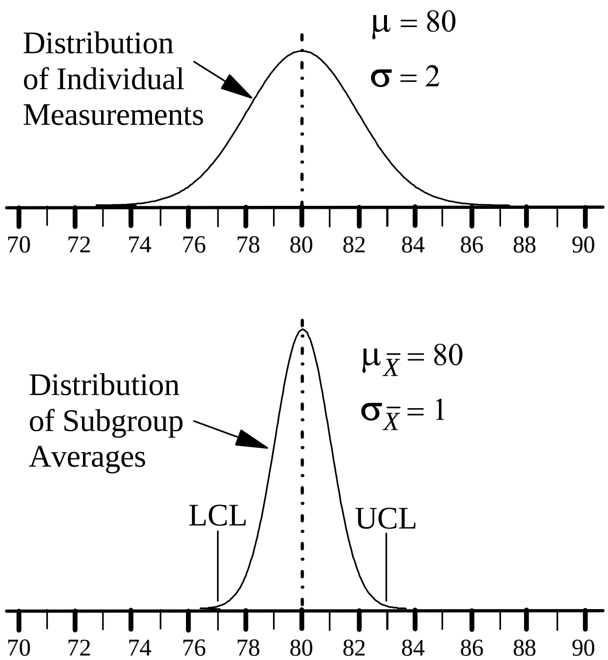

Figure 3 The distribution of subgroup averages.

The standard deviation of the $\overline{X}$ distribution, $\sigma_{\overline{X}}$, is calculated from its relationship to the standard deviation of the individuals and the subgroup size[7].

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{2}{\sqrt{4}} = 1$$

Because the average of the $\overline{X}$ distribution, $\mu_{\overline{X}}$, is equal to μ[7], the $\overline{X}$ s are also centered at 80. With $\sigma_{\overline{X}}$ equal to 1, the *LCL* for the subgroup averages is 77 (80 – 3 × 1), while the *UCL* is 83 (80 + 3 × 1). As long as the process average remains stable, the vast majority of subgroup averages will fall between these control limits and a correct decision will be made to continue producing laminations.

DETECTING A LARGE CHANGE IN μ

The compressibility of the raw material used in making laminations exerts a strong influence on thickness. Each batch of raw material lasts about an hour, with the current batch introduced at 9:30 a.m. Unfortunately, the compressibility of this latest batch has caused the average thickness to leap up to 86, with no change in σ.

The output distribution for thickness after this increase of six units, which translates into a 3σ shift in μ (6 = 3 × 2), is displayed in the top half of Figure 4. With μ only 1σ below the USL of 88 (the initial 4σ distance minus this shift of 3σ), there are now 158,700 *ppm* above this limit instead of the original 32 *ppm*. What is the probability this dramatic decline in quality is detected when the 10:00 a.m. subgroup, the first one collected after this change, is plotted on the control chart?

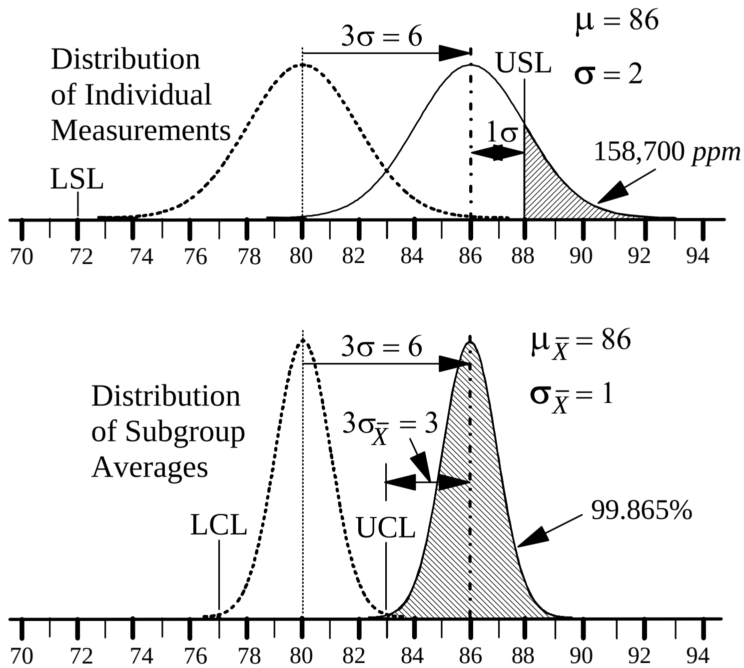

Figure 4 A change in raw material moves μ up to 86.

Because $\mu_{\bar{X}}$ is always equal to μ, when μ shifts by 3σ to 86, so does $\mu_{\bar{X}}$, as is illustrated in the bottom half of Figure 4. $\mu_{\bar{X}}$ is now located three units (86 – 83) above the upper control limit of 83. Equivalently, since $\sigma_{\bar{X}}$ is equal to 1, $\mu_{\bar{X}}$ is $3\sigma_{\bar{X}}$ above the *UCL*.

With all but the area below its lower $3\sigma_{\bar{X}}$ tail (which is .135 percent) above 83, 99.865 percent of the subgroup averages collected from the new process output distribution will be greater than the *UCL*. Plotting one of these on the $\bar{X}$ chart will signal an out-of-control condition by being above the *UCL*. Thus, there is a 99.865 percent chance this 3σ shift in μ is caught by the very next subgroup collected after the change occurs.

After implementing an appropriate corrective action, the average thickness will be restored to its target of 80 and the process will return to producing 64 *ppm*. In addition, having detected an upset at 10:00 a.m., we will sort all laminations manufactured since 9:00 a.m. (the time of the last in-control subgroup) to ensure that customers receive only conforming parts.

DETECTING SMALL MOVEMENTS IN μ

What if the batch of material introduced at 9:30 had caused a change of only three units (a 1.5σ shift) in μ, as indicated in the top half of Figure 5? When μ is bumped up to 83, $\mu_{\overline{X}}$ mimics this move and also increases to 83, as is seen in the bottom half of Figure 5. Because the $\overline{X}$ distribution is now centered on the *UCL* of 83, one half of the $\overline{X}$ s will be greater than the *UCL* while the other half will be less. This situation means that the 10:00 a.m. subgroup average has a 50 percent chance of signaling an out-of-control condition by being above the *UCL*. But there is also a 50 percent chance this subgroup average will be less than 83 and *not* alert us to the change in μ. If this happens, we will think all is well at 10:00 a.m. and continue producing laminations, without making any adjustments to the process average or undertaking any sorting operations.

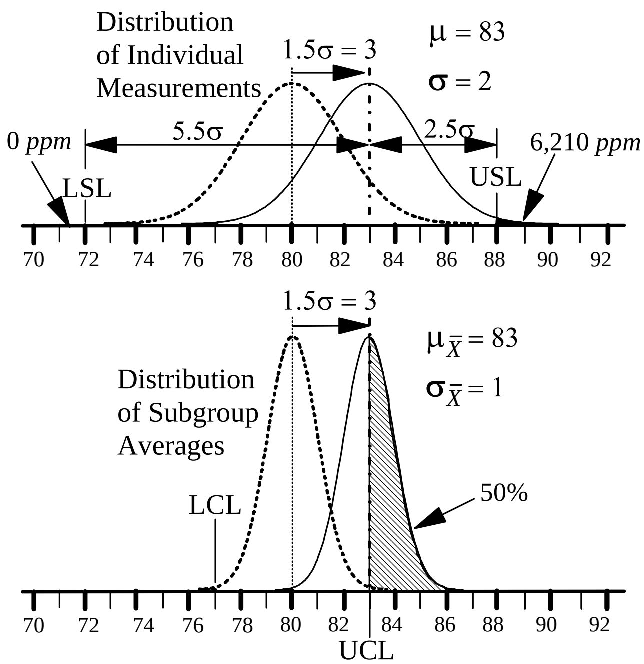

Figure 5 When μ shifts to 83, so does μ of the $\overline{X}$ s.

Suppose the compressibility of the batch of raw material introduced at 10:30 a.m. brings the average thickness back to its target. With μ again equal to 80, the average of the 11:00 a.m. subgroup will probably be in control. Thus, there is a 50 percent chance a 1.5σ shift in μ could begin at 9:30 a.m., adversely affect the process output until 10:30 a.m., then disappear without ever being noticed, even though we are faithfully watching the control chart.

Unfortunately, customers will notice the increase in nonconforming laminations. When μ shifts higher by 1.5σ, instead of being 4.0σ away from each specification, it is now 5.5σ (4.0 + 1.5) above the LSL and only 2.5σ (4.0 – 1.5) below the USL. Under these conditions, there is essentially 0 *ppm* below the LSL but 6210 *ppm* above the USL, for a total of 6210 *ppm* (review Figure 5). This amount is almost ten times more than the 64 *ppm* expected by customers from a process having a C_{PK} reported to be 1.33.

DETECTING MOVEMENTS OF OTHER SIZES

Just as done for shifts of 3.0σ and 1.5σ, probabilities can be calculated for catching shifts of other magnitudes on the next subgroup after the change occurs. A number of these probabilities are listed in Table 1 for several subgroup sizes.

	Subgroup Size		
Shift in μ	3	4	5
0.5σ	.0164	.0228	.0299
1.0σ	.1024	.1587	.2225
1.5σ	.3439	.5000	.6384
2.0σ	.6787	.8413	.9295
2.5σ	.9083	.9772	.9952
3.0σ	.9860	.9986	.9999

Table 1 Probabilities of detecting changes in μ versus subgroup size.

For an $\overline{X}$, *R* chart with *n* equal to four, shifts greater than 1.5σ have more than a 50 percent chance of discovery and should be detected by a diligent operator. However, shifts in μ less than 1.5σ will be caught less than one-half of the time, with the chance of catching a 0.5σ change being only 2 percent. Such low probabilities mean that small perturbations to μ may come and go, without us ever being aware they have negatively impacted our process.

One way to improve the odds of catching small movements in μ is to increase the subgroup size. The chance of detecting a 1.5σ shift increases from 50 percent when *n* is four, to almost 64 percent when *n* is five. However, this chance falls to only 34 percent if *n* is three.

Note that these tabulated probabilities are not for how much or how often μ will shift. Every process is subject to its own unique set of perturbing factors and will experience varying magnitudes and frequencies of transient shifts in μ. But when increasing entropy eventually alters μ by some amount, Table 1 lists the probabilities this change will be promptly detected.

POWER CURVES

The probabilities in Table 1 are plotted on the graph displayed in Figure 6, with one curve for each subgroup size. Called power curves, these lines portray the chances of detecting a shift in μ of a given size (expressed in σ units on the horizontal axis) with the next subgroup collected after the change takes place[8]. For small shifts in μ, all three curves are close to zero. As the size of the shift increases, so does the power of the chart to detect it, with all three curves eventually leveling off close to 100 percent for shifts in excess of 3σ.

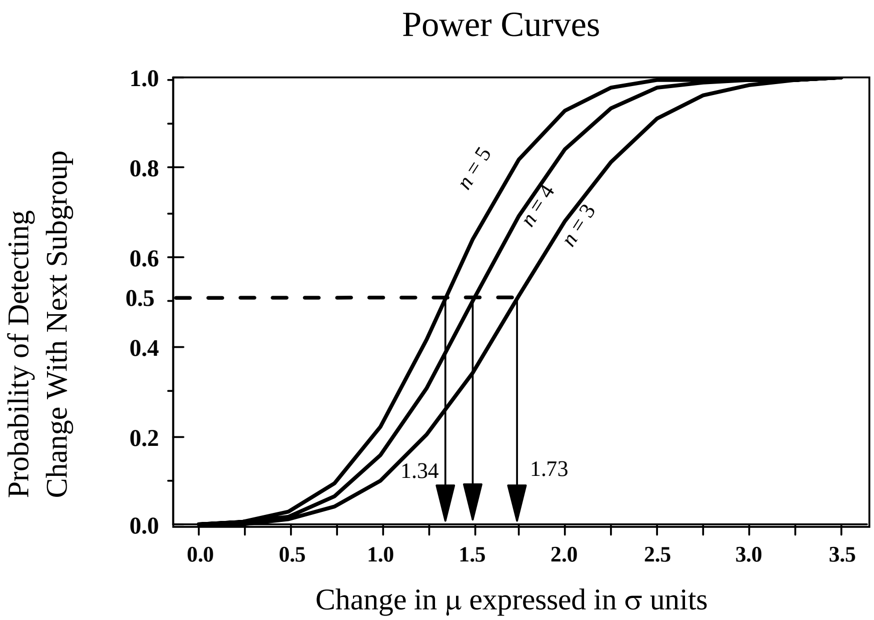

Figure 6 Power curves for subgroup sizes 3, 4, and 5.

The dashed horizontal line drawn on this graph shows there is a 50 percent chance of missing a 1.34σ shift in μ when *n* is five, whereas μ must move by 1.73σ to have this same probability when *n* is only three. Table 2 lists shift sizes that have a 50 percent chance of remaining undetected, called S_{50} values, for subgroup sizes 1 through 6. Temporary movements in μ smaller than $S_{50}\sigma$ are more than likely to be missed by a control chart.

Subgroup Size	S_{50} Value
1	3.00
2	2.12
3	1.73
4	1.50
5	1.34
6	1.22

Table 2 S_{50} values for several subgroup sizes.

For the commonly used subgroup sizes of three, four, and five, these statistically derived adjustment factors range from 1.34σ to 1.73σ. Interestingly, this interval is just slightly larger than the empirically based one of 1.4σ to 1.6σ proposed in the six-sigma literature. However, the six-sigma doctrine applies an identical 1.5σ shift to the average of almost every process, regardless of the subgroup size of its control chart. This rough “rule of thumb” can now be made more precise through the use of Table 2.

MODIFYING THE CAPABILITY ASSESSMENT

As demonstrated above, control charts cannot reliably detect small movements in μ. Even though a chart appears to indicate that a process is in a good state of control, μ may have experienced slight, temporary shifts. Over time, these undetected fluctuations enlarge the spread of the output distribution, as depicted in Figure 7, thereby increasing the amount of nonconforming parts. Although customers receive shipments of parts produced by a process with a dynamic average, most manufacturers estimate outgoing quality assuming a stationary average. It's no wonder few customers believe the quality promises made by these manufacturers.

Acknowledging that a process will experience shifts in μ of various magnitudes, and knowing that not all of these will be discovered, some allowance for them must be made when estimating outgoing quality so customers aren't disappointed. Because shifts ranging in size from 0 up to $S_{50}\sigma$ are the ones likely to remain undetected (larger moves should be caught by the chart), a conservative approach is to assume that every missed shift is as large as $S_{50}\sigma$.

Since shifts can move μ lower or higher, in place of using $\hat{\mu}$ for the process average when estimating capability, $\hat{\mu}$ minus $S_{50}\sigma$ is used to evaluate how well the process output meets the LSL while $\hat{\mu}$ plus $S_{50}\sigma$ is used for determining conformance to the USL. Both of these adjustments are incorporated into the C_{PK} formula, now called the "dynamic" C_{PK} index, by making the following modifications:

$$\begin{aligned}\text{Dynamic}\,\hat{C}_{PK} &= \text{Minimum}\left[\frac{(\hat{\mu}-S_{50}\hat{\sigma})-\text{LSL}}{3\hat{\sigma}},\frac{\text{USL}-(\hat{\mu}+S_{50}\hat{\sigma})}{3\hat{\sigma}}\right]\\ &= \text{Minimum}\left(\frac{\hat{\mu}-\text{LSL}-S_{50}\hat{\sigma}}{3\hat{\sigma}},\frac{\text{USL}-\hat{\mu}-S_{50}\hat{\sigma}}{3\hat{\sigma}}\right)\end{aligned}$$

With μ estimated as 80 and σ as 2 from the control chart in Figure 1, C_{PK} was estimated as 1.33 under the assumption of a stationary average. The associated amount of nonconforming laminations was 64 *ppm* (review Figure 2). By including an adjustment in this assessment for undetected shifts in μ, the estimate of capability will decrease and the expected total number of nonconforming parts will increase.

From Table 2, S_{50} is 1.50 when *n* equals four. Factoring in the possibility of missing shifts in μ of up to 1.50σ drops the C_{PK} index for thickness from 1.33 to only 0.83.

$$\begin{aligned}\text{Dynamic}\,\hat{C}_{PK} &= \text{Minimum}\left(\frac{\hat{\mu}-\text{LSL}-1.50\hat{\sigma}}{3\hat{\sigma}},\frac{\text{USL}-\hat{\mu}-1.50\hat{\sigma}}{3\hat{\sigma}}\right)\\ &= \text{Minimum}\left(\frac{80-72-1.50(2)}{3(2)},\frac{88-80-1.50(2)}{3(2)}\right)\\ &= \text{Minimum}(0.83\,,\,0.83) = 0.83\end{aligned}$$

INTERPRETING THE DYNAMIC C_{PK} INDEX

When μ equals the center of the tolerance, a C_{PK} of 0.83 normally means that μ is 2.5σ (3×0.83) away from either specification limit. Therefore, about 6,210 *ppm* would be below the LSL and another 6,210 *ppm* above the USL, for a total of 12,420 *ppm*.

However, if an undetected change pushes the average thickness 1.5σ higher, there will be 6,210 *ppm* above the USL, but practically no nonconforming parts below the LSL. This is because μ is now 5.5σ ($4.0\sigma + 1.5\sigma$) above the LSL, as illustrated in the top portion of Figure 5. Had μ shifted 1.5σ lower, there would have been 6,210 *ppm* below the LSL, but essentially zero *ppm* exceeding the USL since μ would be 5.5σ away from this upper limit.

Even though μ experiences transient upward and downward shifts over time, it can move in only one direction at a time. There could be 6,210 *ppm* above the USL, or 6,210 *ppm* below the LSL, but not both simultaneously. Therefore, customers should receive no more than 6,210 *ppm* from a process having a dynamic C_{PK} index of 0.83.

This *ppm* level assumes undetected shifts are happening almost constantly and that every one is equal to $S_{50}\sigma$. If a particular process is fairly robust, meaning that μ doesn't shift very often or by very much, customers would receive correspondingly less nonconforming product. In fact, when μ is stationary, customers will get the 64 *ppm* predicted by the conventional C_{PK} index of 1.33.

Thus, depending on the degree of mobility in μ, the actual outgoing quality level will be somewhere between 64 and 6,210 *ppm*. By proclaiming the 6,210 *ppm* level to all customers, the manufacturer should be able to keep its quality promise for this process.

Note that the dynamic C_{PK} is related to the conventional C_{PK} index in the following manner:

$$\begin{aligned}
\text{Dynamic}\, C_{PK} &= \text{Minimum}\left(\frac{\mu - \text{LSL} - S_{50}\sigma}{3\sigma}, \frac{\text{USL} - \mu - S_{50}\sigma}{3\sigma}\right) \\
&= \text{Minimum}\left(\frac{\mu - \text{LSL}}{3\sigma} - \frac{S_{50}\sigma}{3\sigma}, \frac{\text{USL} - \mu}{3\sigma} - \frac{S_{50}\sigma}{3\sigma}\right) \\
&= \text{Minimum}\left(\frac{\mu - \text{LSL}}{3\sigma}, \frac{\text{USL} - \mu}{3\sigma}\right) - \frac{S_{50}\sigma}{3\sigma} \\
&= C_{PK} - \frac{S_{50}}{3}
\end{aligned}$$

Suppose a company desires an outgoing quality level for a particular process to be equivalent to that associated with a dynamic C_{PK} index of 1.67. Knowing that the subgroup size chosen to monitor this process will be five (meaning S_{50} is 1.34), the quality level as measured by a traditional capability study must be such that the conventional C_{PK} index is at least 2.12.

$$\text{Dynamic } C_{PK} = C_{PK} - \frac{S_{50}}{3}$$

$$1.67 = C_{PK} - \frac{1.34}{3}$$

$$2.12 = C_{PK}$$

CHANGING THE SUBGROUP SIZE

By increasing *n*, shifts in μ have a higher probability of detection. With fewer missed changes, the process spread will not expand as much and outgoing quality should be better. For example, if *n* for the chart monitoring thickness had been six instead of four, the S_{50} factor in the dynamic C_{PK} formula would be 1.22 instead of 1.50.

$$\text{Dynamic } \hat{C}_{PK} = \text{Minimum}\left(\frac{\hat{\mu} - \text{LSL} - 1.22\hat{\sigma}}{3\hat{\sigma}}, \frac{\text{USL} - \hat{\mu} - 1.22\hat{\sigma}}{3\hat{\sigma}}\right)$$

$$= \text{Minimum}\left(\frac{80 - 72 - 1.22(2)}{3(2)}, \frac{88 - 80 - 1.22(2)}{3(2)}\right)$$

$$= \text{Minimum}(0.93\ , 0.93) = 0.93$$

Enlarging *n* by two increases the dynamic C_{PK} index from 0.83 to 0.93 and reduces the worst-case expected *ppm* amount by more than half, from 6,210 to just 2,635.

REMAINING QUESTIONS

This article has provided the statistical rationale for adjusting estimates of process capability by including a shift in μ. The range of these statistically derived adjustments is very similar to the one based on the various empirical studies referenced in the six-sigma literature.

However, the dynamic C_{PK} index assumes σ remains stable when μ moves. What if σ is also subjected to undetected increases and decreases? Further studies are needed to determine how these changes would affect estimates of outgoing quality.

In addition, lamination thickness has a bilateral tolerance. Do features with unilateral tolerances need to be handled differently regarding the shift in μ? Also, the output for thickness was assumed to be close to a normal distribution. What effects do shifts in μ and σ have on capability estimates when the process output has a non-normal distribution, especially a nonsymmetrical one?

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